One-dimensional measles dynamics

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Abstract

The SEIR model for the transmission dynamics of measles is extended by the addition of second-order space derivatives to enable the geographic spread of the disease in a population which has not been vaccinated against it.

The resulting system of three reaction-diffusion equations is solved by a convergent finite-difference technique which is second-order accurate in space and time. A parallel implementation procedure is studied and the method is tested using two initial distributions.

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1. Introduction

Measles is still a cause of death in children in underdeveloped countries, where they are likely to be already suffering from malnutrition and poor health. The disease is also highly contagious in monkeys, the only other known host in which measles develops spontaneously.

Measles is a disease of all climates and races, and susceptibility is universal. It must have been common in the ancient world, but no accurate account occurs in history until the classical description by Rhazes in A.D. 915. Thomas Sydenham, in the 17th century, clearly distinguished it from scarlet fever, with
which it had been confused. P.L. Panum, in 1847, published the results of his study of a measles epidemic in the Faroe Islands, definitely establishing the incubation and infectivity periods of measles, see [16].

2. Mathematical epidemiology

The application of mathematics to the study of infectious diseases appears to have been initiated by Daniel Bernoulli in 1760 [3]. He used a mathematical method to evaluate the effectiveness of the techniques of variolation against smallpox, with the aim of influencing public health policy. There then followed a long gap until the middle of the 19th century when, in 1840, William Farr effectively fitted a normal curve to smoothed quarterly data on deaths from smallpox in England and Wales over the period 1837–1839. This descriptive approach was developed further by Brownlee [8] who fitted a Pearsonian frequency distribution curve to a large series of epidemics. The empirical approaches adopted by Farr and Brownlee [3] were in great contrast to the work of two other scientists of the same period, Hamer and Ross. Their contribution was to apply post-germ-theory-thinking towards the solution of two specific quantitative problems: the regular occurrence of measles epidemics and the relationship between numbers of mosquitoes and the incidence of malaria [13,17,20]. They were the first to formulate specific theories about the transmission of infectious diseases in simple but precise mathematical statements and to investigate the properties of the resulting models. Their work, in conjunction with the studies of Ross and Hudson [21], Soper [25], and Kermack and McKendrick [14] began to provide a firm theoretical framework for the investigation of observed patterns.

Hamer [13] postulated that the course of an epidemic depends on the rate of contact between susceptible and infectious individuals. This notion has become one of the most important concepts in mathematical epidemiology; it is the so-called mass action principle in which the net rate of spread of infection is assumed to be proportional to the product of the density of susceptible people multiplied by the density of infectious individuals. The principle was originally formulated in a discrete-time model, but in 1908 Ross [20] (celebrated as the discoverer of malaria transmission by mosquitoes) translated the problem into a continuous-time framework in his pioneering work on the dynamics of malaria (see, also, [3]).

The ideas of Hamer and Ross were extended and explored in more detail by Soper [25] who deduced the underlying mechanisms responsible for the often-observed periodicity of epidemics, and by Kermack and McKendrick [14] who established the celebrated threshold theory. This theory, according to which the introduction of a few infectious individuals into a community of susceptibles will not give rise to an epidemic outbreak unless the density or number of
susceptibles is above a certain critical value, is, in conjunction with the mass action principle, a cornerstone of modern theoretical epidemiology [3].

Since this early beginning, the growth in the literature concerned with mathematical epidemiology has been very rapid indeed. Recent reviews of the literature have been published by Bailey [5], Bolker and Grenfell [7], Dietz [9], Dietz and Schenzle [11], Schenzle [22] and Tidd et al. [26]. In particular, models incorporating seasonality [4,12,15,19] and age structure [2,10,22] generate important predictions both about the likely performance of vaccination strategies and the observed dynamics of infection.

In this paper, a spatially-structured (reaction-diffusion) model of measles will be studied. Recent extensions of the SEIR model have included heterogeneities in terms of age [2,11,22,27], seasonality [4,22–24], and spatial structure [6,18,23].

3. The reaction-diffusion system

The partial differential equation (PDE) model which encompasses a variety of the models in the literature and gives a formal structure for the reaction-diffusion system is given below. In order to proceed, the epidemic is assumed to diffuse through space. Also, it is assumed that all births are into the susceptible class, and that births exactly balance deaths so that the total population size, \( N \), is constant.

The reaction-diffusion equations are given by (see, also, [1])

\[
\begin{align*}
\frac{\partial S}{\partial t} &= \mu N - (\mu + \beta I)S + \alpha \frac{\partial^2 S}{\partial x^2}, \\
\frac{\partial E}{\partial t} &= \beta IS - (\mu + \sigma)E + \alpha \frac{\partial^2 E}{\partial x^2}, \\
\frac{\partial I}{\partial t} &= \sigma E - (\mu + \gamma) + \alpha \frac{\partial^2 I}{\partial x^2}
\end{align*}
\] (3.1)

in which \( S = S(x,t) \), \( E = E(x,t) \) and \( I = I(x,t) \) are the number of susceptibles, exposed and infectious individuals, respectively, at time \( t \) and distance \( x \) from the origin \((-L < x < L)\); \( \alpha > 0 \) is the diffusion rate.

The initial conditions are of the form

\[
S(x,0) = S^0(x), \quad E(x,0) = E^0(x), \quad I(x,0) = I^0(x); \quad -L \leq x \leq L \quad (3.2)
\]

and the boundary conditions are

\[
\begin{align*}
\frac{\partial S(\pm L,t)}{\partial x} &= \frac{\partial E(\pm L,t)}{\partial x} = \frac{\partial I(\pm L,t)}{\partial x} = 0; \quad t > 0.
\end{align*}
\] (3.3)
On differentiating with respect to $t$, the equations in (3.1) give

$$
S_t - \alpha S_{xt} + (\mu + \beta I)S_t + \beta I_tS = 0,
$$

$$
E_t - \alpha E_{xt} + (\mu + \sigma)E_t - \beta IS_t - \beta IS = 0,
$$

$$
I_t - \alpha I_{xt} + (\mu + \gamma)I_t - \sigma E_t = 0.
$$

(3.4)

It will be assumed that the PDEs in (3.1) are defined for $-L < x < L$, $t > 0$ and so, for these ranges, the initial/boundary-value problem (IBVP) ((3.1)–(3.3)) is symmetric about the line $x = 0$. This may be exploited in deriving a numerical method by taking

$$
S_x(0,t) = E_x(0,t) = I_x(0,t) = 0; \quad t > 0,
$$

$$
S_x(L,t) = E_x(L,t) = I_x(L,t) = 0; \quad t > 0,
$$

(3.5)

as the boundary conditions and equations (3.2) as the initial conditions for $0 \leq x \leq L$ and solving the PDEs in (3.1) for $0 \leq x \leq L$, $t > 0$ subject to (3.2) and (3.5).

The terms $1/\mu$, $1/\sigma$, $1/\gamma$ are the average life expectancy, disease incubation and infectious period, respectively; $\beta$ denotes the infection rate. It is assumed that the incubation period coincides with the latent period. All time-related parameters are measured in years. The SEIR model will be considered for the following set of parameter values

$$
N = 5 \times 10^7,
$$

$$
\mu = 0.022 \text{ years}^{-1},
$$

$$
\sigma = 45.6 \text{ years}^{-1},
$$

$$
\gamma = 73.0 \text{ years}^{-1}.
$$

(3.6)

These values represent a population size of 50 million, average life expectancy of 50 years and incubation and infectious periods of roughly eight and five days, respectively, as considered by Bolker and Grenfell [7].

4. Discretization and notations

The interval $0 \leq x \leq L$ is divided into $M + 1$ subintervals each of width $h$ so that $(M + 1)h = L$ and the time interval $t \geq 0$ is discretized in steps of length $\ell$. The open region $\Omega = [0 < x < L] \times [t > 0]$ and its boundary $\partial \Omega$ consisting of the lines $x = 0$, $x = L$ and $t = 0$ have thus been covered by a rectangular mesh having coordinates of the form $(x_m, t_n)$ where $x_m = mh$ ($m = 0, 1, 2, \ldots, M$, $M + 1$) and $t_n = n\ell$ ($n = 0, 1, 2, \ldots$).
The solutions of (3.1), (3.2) and (3.5) at the typical mesh point \((x_m, t_n)\) are, of course, \(S(x_m, t_n)\), \(E(x_m, t_n)\) and \(I(x_m, t_n)\): these may be denoted by \(S^n_m\), \(E^n_m\) and \(I^n_m\), respectively. The theoretical solutions of numerical approximations to (3.1) at the same mesh point will be denoted by \(A^n_m\), \(B^n_m\) and \(C^n_m\), respectively, while the values actually obtained, which may be subject, for example, to round-off errors, will be denoted by \(f^n_m\), \(g^n_m\) and \(h^n_m\), respectively.

5. Numerical methods

5.1. Numerical method for \(S\)

Finite-difference methods are developed by approximating the time derivative in the first equation in (3.1) by the first-order forward-difference replacement

\[
S_t(x, t) \approx \frac{S(x, t + \ell) - S(x, t)}{\ell}
\]

and the space derivative by the second-order approximant

\[
S_{xx} \approx \frac{1}{2} h^{-2} \left\{ S(x - h, t + \ell) - 2S(x, t + \ell) + S(x + h, t + \ell) \right\}
+ \{ S(x - h, t) - 2S(x, t) + S(x + h, t) \},
\]

in which \(x = x_m\) (\(m = 0, 1, 2, \ldots, M, M + 1\)) and \(t = t_n\) (\(n = 0, 1, 2, \ldots\)).

Now, using Eqs. (5.1) and (5.2) in (3.1) and approximating as follows

\[
\frac{A^{n+1}_m - A^n_m}{\ell} + (\mu + \beta C^n_m) A^n_m - \frac{\alpha}{2h^2} \left\{ \{ A^{n+1}_{m-1} - 2A^n_m + A^{n+1}_{m+1} \} \right\} = \mu N = 0,
\]

(5.3)

gives one numerical method while using Eqs. (5.1) and (5.2) in (3.1) and approximating as follows

\[
\frac{A^{n+1}_m - A^n_m}{\ell} + (\mu + \beta C^{n+1}_m) A^n_m - \frac{\alpha}{2h^2} \left\{ \{ A^{n+1}_{m-1} - 2A^n_m + A^{n+1}_{m+1} \} \right\} + \{ A^n_{m-1} - 2A^n_m + A^n_{m+1} \} = \mu N = 0,
\]

(5.4)
gives an alternative numerical method.

In order to obtain a second-order approximation to \(S(x_m, t_{n+1})\), Eqs. (5.3) and (5.4) should be averaged. This gives
\[
\frac{A_{m}^{n+1} - A_{m}^{n}}{\ell} + \frac{1}{2} (\mu + \beta C_{m}^{n}) A_{m}^{n+1} + \frac{1}{2} (\mu + \beta C_{m+1}^{n}) A_{m}^{n} - \frac{a}{2h^2} (A_{m-1}^{n+1} - 2A_{m}^{n+1} + A_{m+1}^{n+1}) - \frac{a}{2h^2} (A_{m-1}^{n} - 2A_{m}^{n} + A_{m+1}^{n}) - \mu N = 0,
\]

which, after rearranging, becomes

\[
- \frac{1}{2} a p A_{m-1}^{n+1} + \left[ 1 + \frac{a p}{2} (\mu + \beta C_{m}^{n}) \right] A_{m}^{n+1} - \frac{1}{2} a p A_{m+1}^{n+1} = \frac{1}{2} a p A_{m-1}^{n+1} + \left[ 1 - \frac{a p}{2} (\mu + \beta C_{m}^{n+1}) \right] A_{m}^{n+1} + \frac{1}{2} a p A_{m+1}^{n+1} + \ell \mu N,
\]

where \( p = \ell / h^2 \).

Verification of second-order accuracy may be obtained by considering the local truncation error \( \mathcal{L}_S = \mathcal{L}'_S \{ S(x, t), E(x, t), I(x, t) ; h, \ell \} \) associated with (5.6) at the point \((x, t) = (x_m, t_n)\), which may be written down from (5.5): it is

\[
\mathcal{L}_S = S(x, t + \ell) - S(x, t) + \frac{1}{2} (\mu + \beta I(x, t)) S(x, t + \ell) + \frac{1}{2} (\mu + \beta I(x, t)) S(x, t) - \frac{a}{2h^2} \left\{ S(x - h, t + \ell) - 2S(x, t + \ell) + S(x + h, t + \ell) \right\} - \mu N - \left\{ S_t(x, t) + (\mu + \beta I(x, t)) S(x, t) - a S_{xx}(x, t) - \mu N \right\}.
\]

Expanding \( S(x, t + \ell) \), \( S(x \pm h, t + \ell) \), \( S(x \pm h, t) \) and \( I(x, t + \ell) \) in (5.7) as Taylor series about \((x, t)\) leads to

\[
\mathcal{L}_S = \left[ \frac{1}{2} S_t + \frac{1}{2} (\mu + \beta I) S_t + \frac{1}{2} \beta S_{tt} - \frac{a}{2} S_{xx} \right] \ell - \frac{1}{12} a h^2 S_{xxxx} + \left[ \frac{1}{6} S_{tt} + \frac{1}{4} (\mu + \beta I) S_t + \frac{1}{4} \beta S_{tt} - \frac{a}{4} S_{xx} \right] \ell^2 + \cdots
\]

Clearly, the term in \( \ell \) vanishes in (5.8), see the first equation in (3.4), leaving

\[
\mathcal{L}_S = - \frac{1}{12} a h^2 S_{xxxx} + \left[ \frac{1}{6} S_{tt} + \frac{1}{4} (\mu + \beta I) S_t + \frac{1}{4} \beta S_{tt} - \frac{a}{4} S_{xx} \right] \ell^2 + \cdots,
\]

which is \( O(h^2 + \ell^2) \) as \( h, \ell \to 0 \).

The finite-difference method (5.6) may be applied for \( m = 1, 2, \ldots, M \) and \( n = 0, 1, 2, \ldots \). In the case \( m = 0 \) it requires some modification and may be simplified a little when \( m = M + 1 \). Applying (5.6) with \( m = 0 \) introduces the terms \( A_{-1}^{n+1} \) and \( A_{1}^{n+1} \). Now, the points \((x_{-1}, t_{n+1})\) and \((x_{1}, t_{n})\) are outside the grid superimposed on \( \Omega \) and \( \partial \Omega \). The Boundary conditions (3.5), however, give, to
second order, \( A^n_{-1} = A^n_1 \) and \( A^{n+1}_{-1} = A^{n+1}_1 \) so that, for \( m = 0 \), Eq. (5.6) may be modified to give

\[
\left[ 1 + \frac{1}{\ell} \left( \mu + \beta C^n_0 \right) \right] A^{n+1}_0 - x p A^{n+1}_1 + \frac{1}{\ell} \beta C^{n+1}_0 A^n_0 \\
= \left( 1 - \frac{1}{\ell} \mu \right) A^n_0 + x p A^n_1 + \ell \mu N. \tag{5.10}
\]

Now, with \( m = M + 1 \), it follows from (3.5) that \( A^n_{M+2} = A^n_M \) and \( A^{n+1}_{M+2} = A^{n+1}_M \) and so (5.6) may be written as

\[
-x p A^{n+1}_M + \left[ 1 + \frac{1}{\ell} \left( \mu + \beta C^n_{M+1} \right) \right] A^{n+1}_{M+1} + \frac{1}{\ell} \beta C^{n+1}_{M+1} A^n_{M+1} \\
= x p A^n_M + \left( 1 - \frac{1}{\ell} \mu \right) A^n_{M+1} + \ell \mu N. \tag{5.11}
\]

5.2. Numerical method for \( E \)

The time derivative in the second equation in (3.1) is approximated by the first-order forward-difference replacement

\[
E_t \approx \frac{[E(x, t + \ell) - E(x, t)]}{\ell} \tag{5.12}
\]

and the space derivative by the second-order approximant

\[
E_{xx} \approx \frac{1}{2} h^{-2} \left[ \{ E(x - h, t + \ell) - 2E(x, t + \ell) + E(x + h, t + \ell) \} \\
+ \{ E(x - h, t) - 2E(x, t) + E(x + h, t) \} \right], \tag{5.13}
\]

in which \( x = x_m \) \((m = 0, 1, 2, \ldots, M, M + 1)\) and \( t = t_n \) \((n = 0, 1, 2, \ldots)\).

Using Eqs. (5.12) and (5.13) in the second equation in (3.1) and approximating as follows

\[
\frac{B^{n+1}_m - B^n_m}{\ell} - \beta C^n_m A^{n+1}_m + (\mu + \sigma) B^{n+1}_m - \frac{\alpha}{2h^2} [(B^{n+1}_{m-1} - 2B^{n+1}_m + B^{n+1}_{m+1}) \\
+ (B^n_{m-1} - 2B^n_m + B^n_{m+1})] = 0, \tag{5.14}
\]

gives one numerical method while using Eqs. (5.12) and (5.13) in (3.1) and approximating as follows

\[
\frac{B^{n+1}_m - B^n_m}{\ell} - \beta C^{n+1}_m A^n_m + (\mu + \sigma) B^n_m - \frac{\alpha}{2h^2} [(B^{n+1}_{m-1} - 2B^{n+1}_m + B^{n+1}_{m+1}) \\
+ (B^n_{m-1} - 2B^n_m + B^n_{m+1})] = 0, \tag{5.15}
\]

gives a second numerical method. To obtain a second-order approximation to \( E(x_m, t_{n+1}) \), Eqs. (5.14) and (5.15) should be averaged. This gives
\[ \frac{B_{m+1}^n - B_m^n}{\ell} - \frac{1}{2} \beta C_m A_{m+1}^n - \frac{1}{2} \beta C_{m+1} A_m^n + \frac{1}{2} (\mu + \sigma) B_m^{n+1} \]
\[ + \frac{1}{2} (\mu + \sigma) B_m^n - \frac{\alpha}{2h^2} \left[ (B_{m-1}^n - 2B_m^n + B_{m+1}^n) + (B_{m-1}^n - 2B_m^n + B_{m+1}^n) \right] = 0, \]
(5.16)

which, after rearranging, becomes

\[ - \frac{1}{2} \bar{\rho} B_{m-1}^{n+1} + \left[ 1 + \bar{\rho} + \frac{1}{2} \ell (\mu + \sigma) \right] B_m^{n+1} - \frac{1}{2} \bar{\rho} B_{m+1}^{n+1} \]
\[ - \frac{1}{2} \ell \beta C_m A_{m+1}^n - \frac{1}{2} \ell \beta A_m C_{m+1}^n \]
\[ = \frac{1}{2} \bar{\rho} B_m^n + \left[ 1 - \bar{\rho} - \frac{1}{2} \ell (\mu + \sigma) \right] B_m^n + \frac{1}{2} \bar{\rho} B_{m+1}^n, \]
(5.17)

where \( \bar{\rho} = \ell / h^2 \).

The local truncation error \( L_E = L_E[S(x, t), E(x, t), I(x, t); h, \ell] \) associated with (5.17) at the point \((x, t) = (x_m, t_n)\) may be written down from (5.16). Using the second equation in (3.4) reveals that the coefficient of \( \ell \) in \( L_E \) vanishes leaving

\[ L_E = - \frac{1}{12} \bar{\rho} h^2 E_{xxxx} + \left[ \frac{1}{6} E_{ttt} + \frac{1}{4}(\mu + \sigma)E_{tt} \right. \]
\[ - \frac{1}{4} \beta IS_{tt} - \frac{1}{4} \beta SI_{tt} - \frac{1}{4} \bar{\rho} E_{xx} \ell^2 + \cdots, \]
(5.18)

which confirms second-order accuracy as \( h, \ell \to 0 \).

As in the case of \( S \), Eq. (5.17) may be modified for use with \( m = 0 \) and \( m = M + 1 \) using the equations in (3.5) to obtain, respectively,

\[ \left[ 1 + \bar{\rho} + \frac{1}{2} \ell (\mu + \sigma) \right] B_0^{n+1} - \bar{\rho} B_1^{n+1} - \frac{1}{2} \ell \beta C_0 A_0^{n+1} - \frac{1}{2} \ell \beta C_{n+1} A_0^n \]
\[ = \left[ 1 - \bar{\rho} - \frac{1}{2} \ell (\mu + \sigma) \right] B_0^n + \bar{\rho} B_1^n, \]
(5.19)

and

\[ - \bar{\rho} B_{M+1}^{n+1} + \left[ 1 + \bar{\rho} + \frac{1}{2} \ell (\mu + \sigma) \right] B_{M+1}^{n+1} - \frac{1}{2} \ell \beta C_{M+1} A_{M+1}^{n+1} - \frac{1}{2} \ell \beta C_{m+1} A_{M+1}^n \]
\[ = \bar{\rho} B_{M+1}^n + \left[ 1 - \bar{\rho} - \frac{1}{2} \ell (\mu + \sigma) \right] B_{M+1}^n, \]
(5.20)
5.3. Numerical method for $I$

The time derivative in the third equation in (3.1) is approximated by the first-order forward-difference replacement

$$I_t \approx \frac{I(x, t + \ell) - I(x, t)}{\ell}$$  \hspace{1cm} (5.21)

and the space derivative by the second-order approximant

$$I_{xx} \approx \frac{1}{2} h^{-2} \{I(x - h, t + \ell) - 2I(x, t + \ell) + I(x + h, t + \ell)\}
+ \{I(x - h, t) - 2I(x, t) + I(x + h, t)\},$$  \hspace{1cm} (5.22)

in which $x = x_m \ (m = 0, 1, 2, \ldots, M, M + 1)$ and $t = t_n \ (n = 0, 1, 2, \ldots)$. This gives

$$\frac{C_{m+1}^n - C_m^n}{\ell} - \sigma B_{m+1}^n + (\mu + \gamma) C_{m+1}^n - \frac{\alpha}{2h^2} [(C_{m-1}^{n+1} - 2C_m^{n+1} + C_{m+1}^{n+1})
+ (C_{m-1}^n - 2C_m^n + C_{m+1}^n)] = 0, \hspace{1cm} (5.23)$$

as one numerical method for solving for $I$, while approximating as follows

$$\frac{C_{m+1}^n - C_m^n}{\ell} - \sigma B_m^n + (\mu + \gamma) C_m^n - \frac{\alpha}{2h^2} [(C_{m-1}^{n+1} - 2C_m^{n+1} + C_{m+1}^{n+1})
+ (C_{m-1}^n - 2C_m^n + C_{m+1}^n)] = 0, \hspace{1cm} (5.24)$$

gives a second numerical method. In order to obtain a second-order approximation to $I(x_m, t_{n+1})$, Eqs. (5.23) and (5.24) are averaged to obtain

$$\frac{C_{m+1}^n - C_m^n}{\ell} - \frac{1}{2} \sigma B_{m+1}^n - \frac{1}{2} \sigma B_m^n + \frac{1}{2} (\mu + \gamma) C_{m+1}^n + \frac{1}{2} (\mu + \gamma) C_m^n
- \frac{\alpha}{2h^2} [(C_{m-1}^{n+1} - 2C_m^{n+1} + C_{m+1}^{n+1}) + (C_{m-1}^n - 2C_m^n + C_{m+1}^n)] = 0, \hspace{1cm} (5.25)$$

which, after rearranging, becomes

$$- \frac{1}{2} \alpha p C_{m+1}^{n+1} + \left[ 1 + \alpha p + \frac{1}{2} \ell (\mu + \gamma) \right] C_{m+1}^n - \frac{1}{2} \alpha p C_{m+1}^{n+1} - \frac{1}{2} \ell \sigma B_{m+1}^n
= \frac{1}{2} \alpha p C_{m+1}^{n+1} + \left[ 1 - \alpha p - \frac{1}{2} \ell (\mu + \gamma) \right] C_m^n + \frac{1}{2} \alpha p C_{m+1}^n + \frac{1}{2} \ell \sigma B_m^n,$$  \hspace{1cm} (5.26)

where $p = \ell/h^2$.

The local truncation error $L_I = L_I[S(x, t), E(x, t), I(x, t); h, \ell]$ associated with (5.26) at the point $(x, t) = (x_m, t_n)$ may be written down from (5.25). Using the third equation in (3.4) reveals that the coefficient of $\ell$ in ($L_I$) vanishes leaving
\[ L_1 = -\frac{1}{12} \alpha h^2 I_{xxxx} + \left[ \frac{1}{6} I_{tt} + \frac{1}{4} \mu \gamma I_{tt} - \frac{1}{4} \alpha E_{tt} - \frac{1}{4} \alpha I_{xtt} \right] \ell^2 + \cdots, \]  
(5.27)

confirming \( O(h^2 + \ell^2) \) accuracy as \( h, \ell \to 0 \).

Eq. (5.26) may be modified, as in the cases of \( S \) and \( E \), for use with \( m = 0 \) and \( m = M + 1 \) using the boundary conditions in (3.5) to obtain

\[
\left[ 1 + \alpha p + \frac{1}{2} \ell (\mu + \gamma) \right] C_{0}^{n+1} - \alpha p C_{1}^{n+1} - \frac{1}{2} \ell \sigma B_{0}^{n+1} \\
= \left[ 1 - \alpha p - \frac{1}{2} \ell (\mu + \gamma) \right] C_{0}^{n} + \alpha p C_{1}^{n} + \frac{1}{2} \ell \sigma B_{0}^{n},
\]
(5.28)

and

\[
-\alpha p C_{M}^{n+1} + \left[ 1 + \alpha p + \frac{1}{2} \ell (\mu + \gamma) \right] C_{M+1}^{n+1} - \frac{1}{2} \ell \sigma B_{M+1}^{n+1} \\
= \alpha p C_{M}^{n} + \left[ 1 - \alpha p - \frac{1}{2} \ell (\mu + \gamma) \right] C_{M+1}^{n} + \frac{1}{2} \ell \sigma B_{M+1}^{n}.
\]
(5.29)

6. Implementation

Let \( A^{n+1} = \begin{bmatrix} A_{0}^{n+1}, A_{1}^{n+1}, \ldots, A_{M+1}^{n+1} \end{bmatrix}^\top \), \( B^{n+1} = \begin{bmatrix} B_{0}^{n+1}, B_{1}^{n+1}, \ldots, B_{M+1}^{n+1} \end{bmatrix}^\top \) and \( C^{n+1} = \begin{bmatrix} C_{0}^{n+1}, C_{1}^{n+1}, \ldots, C_{M+1}^{n+1} \end{bmatrix}^\top \), where \( T \) denotes transpose. Then the price to be paid in using (5.6), (5.10), (5.11), (5.17), (5.19), (5.20), (5.26), (5.28) and (5.29) to obtain \( O(h^2 + \ell^2) \) solutions to (3.1) as \( h, \ell \to 0 \) is that \( A^{n+1}, B^{n+1} \) and \( C^{n+1} \) cannot be obtained by solving three linear algebraic systems of order \( (M+2) \) at each time step, either in sequence or in parallel. Instead, because of the appearance of the elements of \( C^{n+1} \) in (5.6), (5.10) and (5.11), the elements of \( A^{n+1} \) and \( C^{n+1} \) in (5.17), (5.19) and (5.20) and the elements of \( B^{n+1} \) in (5.26), (5.28) and (5.29), the vectors \( A^{n+1}, B^{n+1} \) and \( C^{n+1} \) must be obtained simultaneously by solving a linear algebraic system of order \( (3M+6) \) at each time step.

Let \( U^{n+1} = \begin{bmatrix} (A^{n+1})^\top, (B^{n+1})^\top, (C^{n+1})^\top \end{bmatrix}^\top \) and \( U^n = \begin{bmatrix} (A^n)^\top, (B^n)^\top, (C^n)^\top \end{bmatrix}^\top \). Then it may be seen that (5.6), (5.10), (5.11), (5.17), (5.19), (5.20), (5.26), (5.28) and (5.29) may be written in matrix-vector form as,

\[
W^n U^{n+1} = M^n U^n + b
\]
(6.1)
in which
\[ W^n = \begin{pmatrix} X^n & O & H^n_1 \\ F^n_1 & Y^n & F^n_2 \\ O & H^n_2 & Z^n \end{pmatrix}, \] (6.2) 

\[ M^n = \begin{pmatrix} P^n & O & O \\ O & Q^n & O \\ O & H^n_3 & R^n \end{pmatrix}, \] (6.3) 

where \( O \) is the zero matrix of order \((M + 2)\) and the vector \( \mathbf{b} \) is a column-vector of order \((3M + 6)\) which is given by 

\[ \mathbf{b} = \begin{bmatrix} \ell \mu N, \ldots, \ell \mu N, 0, \ldots, 0 \end{bmatrix}^T. \] (6.4) 

The matrices \( W^n \) and \( M^n \) are both of order \((3M + 6)\) and their submatrices are of order \((M + 2)\) and are given by 

\[ X^n = \begin{pmatrix} v_0 & -xp \\ -\frac{1}{2} xp & v_1 & -\frac{1}{2} xp \\ & -\frac{1}{2} xp & v_2 & -\frac{1}{2} xp \\ & & \ddots & \ddots & \ddots \\ & & & -\frac{1}{2} xp & v_M & -\frac{1}{2} xp \\ & & & & -xp & v_{M+1} \end{pmatrix}, \] (6.5) 

\[ Y^n = \begin{pmatrix} v_Y & -xp \\ -\frac{1}{2} xp & v_Y & -\frac{1}{2} xp \\ & -\frac{1}{2} xp & v_Y & -\frac{1}{2} xp \\ & & \ddots & \ddots & \ddots \\ & & & -\frac{1}{2} xp & v_Y & -\frac{1}{2} xp \\ & & & & -xp & v_Y \end{pmatrix}, \] (6.6)
\[
Z^n = \begin{pmatrix}
v_Z & -\frac{1}{2} \beta p \\
-\frac{1}{2} \beta p & v_Z & -\frac{1}{2} \beta p \\
& & v_Z & -\frac{1}{2} \beta p \\
& & & \ddots & \ddots & \ddots \\
& & & & -\frac{1}{2} \beta p & v_Z & -\frac{1}{2} \beta p \\
& & & & & -\frac{1}{2} \beta p & v_Z \\
& & & & & & -\frac{1}{2} \beta p
\end{pmatrix}, \quad (6.7)
\]

\[
P^n = \begin{pmatrix}
v_P & -\frac{1}{2} \beta p \\
\frac{1}{2} \beta p & v_P & \frac{1}{2} \beta p \\
& & \frac{1}{2} \beta p & v_P & \frac{1}{2} \beta p \\
& & & \ddots & \ddots & \ddots \\
& & & & \frac{1}{2} \beta p & v_P & \frac{1}{2} \beta p \\
& & & & & \frac{1}{2} \beta p
\end{pmatrix}, \quad (6.8)
\]

\[
Q^n = \begin{pmatrix}
v_Q & \frac{1}{2} \beta p \\
\frac{1}{2} \beta p & v_Q & \frac{1}{2} \beta p \\
& & \frac{1}{2} \beta p & v_Q & \frac{1}{2} \beta p \\
& & & \ddots & \ddots & \ddots \\
& & & & \frac{1}{2} \beta p & v_Q & \frac{1}{2} \beta p \\
& & & & & \frac{1}{2} \beta p
\end{pmatrix}, \quad (6.9)
\]

\[
R^n = \begin{pmatrix}
v_R & \frac{1}{2} \beta p \\
\frac{1}{2} \beta p & v_R & \frac{1}{2} \beta p \\
& & \frac{1}{2} \beta p & v_R & \frac{1}{2} \beta p \\
& & & \ddots & \ddots & \ddots \\
& & & & \frac{1}{2} \beta p & v_R & \frac{1}{2} \beta p \\
& & & & & \frac{1}{2} \beta p
\end{pmatrix}, \quad (6.10)
\]
\[ H_1^n = \text{diag}\left\{ \frac{1}{2} \ell \beta A_m^n \right\}, \quad H_2^n = \text{diag}\left\{ -\frac{1}{2} \ell \sigma I_{(M+2)} \right\}, \]
\[ H_3^n = \text{diag}\left\{ \frac{1}{2} \ell \sigma I_{(M+2)} \right\}, \quad F_1^n = \text{diag}\left\{ -\frac{1}{2} \ell \beta C_m^n \right\} \]
\[ F_2^n = \text{diag}\left\{ -\frac{1}{2} \ell \beta A_m^n \right\}, \]

(6.11)

where
\[ v_i = 1 + x p + \frac{1}{2} \ell (\mu + \beta C_m^n), \quad i = 0, 1, 2, \ldots, M, M + 1, \]
\[ v_Y = 1 + x p + \frac{1}{2} \ell (\mu + \sigma), \]
\[ v_Z = 1 + x p + \frac{1}{2} \ell (\mu + \gamma), \]
\[ v_P = 1 - x p - \frac{1}{2} \ell \mu, \]
\[ v_Q = 1 - x p - \frac{1}{2} \ell (\mu + \sigma), \]
\[ v_R = 1 - x p - \frac{1}{2} \ell (\mu + \gamma), \]

\( I_{(M+2)} \) is the identity matrix of order \((M + 2)\) and \( p = \ell / h^2 \).

Further research reveals that it is not necessary to compute \( A^{n+1}, B^{n+1} \) and \( C^{n+1} \) simultaneously by solving the linear algebraic system (6.1), which is of order \((3N + 6)\), on a single processor. It is possible after all to compute \( A^{n+1}, B^{n+1} \) and \( C^{n+1} \) in parallel on an architecture with three processors; this is made possible because \( F_1^n, X^n Y^n, F_2^n, Z^n, X^n Z^n F_1^n, F_2^n, \) and \( Y^n \) commute with \( X^n, H_1^n, \]
\( Z^n, H_1^n, F_1^n Z^n X^n, H_1^n \) and \( H_2^n \), respectively.

Eq. (6.1) may be split to give the following equations
\[ X^n A^{n+1} + H_1^n C^{n+1} = P^n A^n + b, \]
\[ F_1^n A^{n+1} + Y^n B^{n+1} + F_2^n C^{n+1} = Q^n B^n, \]
\[ H_2^n B^{n+1} + Z^n C^{n+1} = H_3^n B^n + R^n C^n \]

(6.13)

which may be solved for \( A^{n+1}, B^{n+1} \) and \( C^{n+1} \). These vectors may then be obtained using an architecture with three processors on which to implement the following algorithm.

**Algorithm 1.** Processor 1: Solve
\[
\begin{bmatrix}
Y^n Z^n X^n - H_2^n (F_1^n X^n - H_1^n F_1^n)
\end{bmatrix} A^{n+1}
= \begin{bmatrix}
Y^n Z^n - H_2^n F_2^n
\end{bmatrix} P^n A^n + \begin{bmatrix}
Y^n Z^n - H_2^n F_2^n
\end{bmatrix} b
+ \begin{bmatrix}
H_2^n H_1^n Q^n - Y^n H_1^n H_3^n
\end{bmatrix} B^n - Y^n H_1^n R^n C^n
\]
for \( A^{n+1} \).
Processor 2: Solve

\[
\begin{align*}
&\left[ X^n (Z^n Y^n - F^n_2 H^n_2) + F^n_1 H^n_1 H^n_3 \right] B^{n+1} \\
&= \left[ X^n (Z^n Q^n - F^n_2 H^n_3) + F^n_1 H^n_1 H^n_3 \right] B^n + \left[ F^n_1 H^n_1 - X^n F^n_2 \right] R^n C^n \\
&\quad - F^n_1 Z^n P^n A^n - F^n_1 Z^n b
\end{align*}
\]

for \( B^{n+1} \).

Processor 3: Solve

\[
\begin{align*}
&\left[ H^n_2 (X^n F^n_2 - F^n_1 H^n_1) - X^n Y^n Z^n \right] C^{n+1} \\
&= \left[ H^n_2 X^n Q^n - X^n Y^n H^n_3 \right] B^n - H^n_2 F^n_1 P^n A^n - X^n Y^n R^n C^n - H^n_2 F^n_1 b
\end{align*}
\]

for \( C^{n+1} \).

All the three processors solve a linear algebraic system of order \((M + 2)\) at each time step.

7. Convergence

The von Neumann and matrix methods for analysing stability both failed to give a criterion for the stability of the methods developed in this paper. The maximum-principle method, on the other hand, by assuming the boundedness of the derivatives \( \partial^2 S/\partial x^2 \), \( \partial^2 E/\partial x^2 \), \( \partial^4 I/\partial x^4 \), \( \partial^4 E/\partial x^4 \), and \( \partial^4 I/\partial x^4 \), in the region \( 0 \leq x \leq L \), \( 0 \leq t \leq T \), and the boundedness of \( S^n_{m+1} - S^n_m \), \( E^n_{m+1} - E^n_m \) and \( I^n_{m+1} - I^n_m \) in the same region, confirms that, for the method developed in Section 5, \( A^n_{m+1} \) converges uniformly to \( S^n_{m+1} \), \( B^n_{m+1} \) converges uniformly to \( E^n_{m+1} \) and \( C^n_{m+1} \) converges uniformly to \( I^n_{m+1} \). Full details of this complicated analysis are given in \([1,28]\).

8. Numerical results and discussions

To test the second-order methods (5.6), (5.10), (5.11), (5.17), (5.19), (5.20), (5.26), (5.28) and (5.29) for susceptibles, exposed and infectives, respectively, the initial/boundary-value problem consisting of the Eqs. (3.1), (3.2) and (3.5) was solved using the set of parameters given in (3.6) for \( N \), \( \mu \), \( \sigma \) and \( \gamma \) with the infection rate, \( \beta \), chosen to be \( \beta = 5 \times 10^{-4} \) and the diffusion rate, \( \alpha \), to be \( \alpha = 0.01 \).

In the following numerical experiments the total numbers of susceptibles, exposed and infected individuals are taken to be \( 1.25 \times 10^7 \), \( 5 \times 10^4 \) and \( 3 \times 10^4 \), respectively.
respectively. The ways in which each is distributed over the interval \(0 \leq x \leq 1\) give the functions \(S^0(x)\), \(E^0(x)\) and \(I^0(x)\) in (3.2).

8.1. Experiment A

In this experiment, hat-shaped initial distributions are used for \(S\), \(E\) and \(I\). Taking \(h = 0.025\) so that \(M = 39\), giving the discretization \(x_i \ (i = 0, 1, \ldots, 40)\) of the interval \(0 \leq x \leq 1\), the initial conditions in (3.2) are distributed as follows (see Figs. 1 and 2)

\[
S(x_i, 0) = \begin{cases} 
31250i, & 0 \leq i \leq \frac{M + 1}{2}, \\
31250(M + 1 - i), & \frac{M + 1}{2} < i \leq M + 1,
\end{cases}
\]

Fig. 1. Experiment A, initial distributions of susceptibles, exposed and infectives.

Fig. 2. Experiment A, initial distributions of susceptibles (—), exposed (•••) and infectives (---).
Initially, the maximum value of each class of individuals is concentrated at the middle of the interval $0 \leq x \leq 1$ and the numbers decrease linearly to zero at the boundaries $x = 0$ and $x = 1$.

As time increases, the number of susceptibles decreases whereas the numbers of both exposed and infectious individuals increase until the time $t = 0.09$ after which the number of susceptibles becomes less than the number of exposed individuals, near the middle of the interval (see Fig. 3). This reveals the dynamic behaviour of measles and is as would be expected.

Fig. 3 shows the distribution of susceptibles, exposed and infectious individuals at time $t = 0.1$. Figs. 4–7, respectively, give three-dimensional plots of susceptible, exposed, infectious and recovered individuals for $0 \leq x \leq 1$ and $0 \leq t \leq 0.1$. The profiles in Fig. 3 can be seen clearly in Figs. 4–6 by locating the plane $t = 0.1$ in each figure.

As the diffusion rate, $\alpha$, increases, the dynamic behaviour of measles changes as shown in Figs. 8–13; as the diffusion rate, $\alpha$, is increased, the number of susceptibles becomes larger than both the numbers of exposed and infected individuals and both the exposed and infected individuals spread on the $x$-axis.

![Fig. 3. Experiment A, dynamics of measles at time $t = 0.1$, $\alpha = 0.01$, $\ell = 0.001$ and $h = 0.025$; susceptibles (---), exposed (---) and infectives (---).](image-url)
Fig. 4. Experiment A, three-dimensional distribution of susceptibles; $\ell = 0.001$ and $h = 0.025$.

Fig. 5. Experiment A, three-dimensional distribution of exposed individuals; $\ell = 0.001$ and $h = 0.025$.

Fig. 6. Experiment A, three-dimensional distribution of infectives; $\ell = 0.001$ and $h = 0.025$. 
Fig. 7. Experiment A, three-dimensional distribution of recovered individuals; $\ell = 0.001$ and $h = 0.025$.

Fig. 8. Experiment A, dynamics of measles at time $t = 0.1$, $x = 0.0001$, $\ell = 0.001$ and $h = 0.025$; susceptibles (---), exposed (-----) and infectives (----).

Fig. 9. Experiment A, dynamics of measles at time $t = 0.1$, $x = 0.001$, $\ell = 0.001$ and $h = 0.025$; susceptibles (---), exposed (-----) and infectives (----).
Fig. 10. Experiment A, dynamics of measles at time $t = 0.1$, $x = 0.03$, $\ell = 0.001$ and $h = 0.025$; susceptibles (—), exposed (---) and infectives (···).

Fig. 11. Experiment A, dynamics of measles at time $t = 0.1$, $x = 0.04$, $\ell = 0.001$ and $h = 0.025$; susceptibles (—), exposed (---) and infectives (···).

Fig. 12. Experiment A, dynamics of measles at time $t = 0.1$, $x = 0.05$, $\ell = 0.001$ and $h = 0.025$; susceptibles (—), exposed (---) and infectives (···).
In this experiment, $h$ is chosen to be 0.05 so that $M = 19$ and the initial conditions are distributed as in Figs. 14 and 15; they are of the form

$$S(x_i,0) = \begin{cases} 
569048, & i = \frac{M+1}{2}, \\
589048, & i = \frac{M-3}{2}, \frac{M-1}{2}, \frac{M+3}{2}, \frac{M+5}{2}, \\
599048, & 0 \leq i \leq \frac{M-5}{2} \text{ and } \frac{M+7}{2} \leq i \leq M+1,
\end{cases}$$

$$E(x_i,0) = \begin{cases} 
10^4, & \frac{M-3}{2} \leq i \leq \frac{M+5}{2}, \\
0, & 0 \leq i < \frac{M-3}{2} \text{ and } \frac{M+5}{2} < i \leq M+1,
\end{cases}$$

Fig. 13. Experiment A, dynamics of measles at time $t = 0.1$, $x = 0.09$, $\ell = 0.001$ and $h = 0.025$; susceptibles (—), exposed (---) and infectives (••).  

8.2. Experiment B

In this experiment, $h$ is chosen to be 0.05 so that $M = 19$ and the initial conditions are distributed as in Figs. 14 and 15; they are of the form

$$S(x_i,0) = \begin{cases} 
569048, & i = \frac{M+1}{2}, \\
589048, & i = \frac{M-3}{2}, \frac{M-1}{2}, \frac{M+3}{2}, \frac{M+5}{2}, \\
599048, & 0 \leq i \leq \frac{M-5}{2} \text{ and } \frac{M+7}{2} \leq i \leq M+1,
\end{cases}$$

$$E(x_i,0) = \begin{cases} 
10^4, & \frac{M-3}{2} \leq i \leq \frac{M+5}{2}, \\
0, & 0 \leq i < \frac{M-3}{2} \text{ and } \frac{M+5}{2} < i \leq M+1,
\end{cases}$$

Fig. 14. Experiment B, initial distributions of susceptibles, exposed and infectives.
where the exposed and infectious individuals are concentrated in the middle of the interval \((0 \leq x \leq 1)\) and the susceptibles are distributed along the whole interval such that the number of susceptible individuals in the middle of the interval is less than the other parts of the interval.

As in experiment A, the number of susceptibles is seen to decrease and those of exposed and infectious individuals are increased as time is increased. This behaviour continues until time \(t = 0.06\) after which the numbers of exposed and infectious individuals become greater than that of susceptibles, as may be expected from the dynamics of the disease. The profiles of the three classes of individual, as predicted by the model, at time \(t = 0.1\) are shown in Fig. 16. Three-dimensional plots of susceptible, exposed, infectious and recovered individuals are shown, respectively, in Figs. 17–20.

As the diffusion rate, \(\alpha\), increases, the number of susceptibles decreases and the numbers of exposed and infected individuals increase and take a larger area on the \(x\)-axis (see Figs. 21–24).

From these experiments, it is seen that the dynamic behaviour of measles depends on the initial distributions and the diffusion rate.
Fig. 16. Experiment B, distribution of susceptibles (—), exposed (−−−) and infectives (···) after 100 iterations (t = 0.1); ℓ = 0.001 and h = 0.05.

Fig. 17. Experiment B, three-dimensional distribution of susceptibles; ℓ = 0.001 and h = 0.05.

Fig. 18. Experiment B, three-dimensional distribution of exposed; ℓ = 0.001 and h = 0.05.
Fig. 19. Experiment B, three-dimensional distribution of infectives; $\ell = 0.001$ and $h = 0.05$.

Fig. 20. Experiment B, three-dimensional distribution of recovered; $\ell = 0.001$ and $h = 0.05$.

Fig. 21. Experiment B, dynamics of measles at time $t = 0.1$, $\alpha = 0.03$, $\ell = 0.001$ and $h = 0.05$; susceptibles (---), exposed (--- -) and infectives (---).
Fig. 22. Experiment B, dynamics of measles at time $t = 0.1$, $x = 0.04$, $\ell = 0.001$ and $h = 0.05$; susceptibles (---), exposed (---) and infectives (---).

Fig. 23. Experiment B, dynamics of measles at time $t = 0.1$, $x = 0.05$, $\ell = 0.001$ and $h = 0.05$; susceptibles (---), exposed (---) and infectives (---).

Fig. 24. Experiment B, dynamics of measles at time $t = 0.1$, $x = 0.09$, $\ell = 0.001$ and $h = 0.05$; susceptibles (---), exposed (---) and infectives (---).
9. Conclusion

A second-order, finite-difference scheme has been developed and implemented in this paper for computing the solutions of the SEIR measles model in one dimension (3.1). A stability analysis revealed that both the von Neumann and matrix stability methods failed to give a criterion for the stability of the method so the maximum-principle analysis was used to show that the scheme is convergent. Two numerical experiments were chosen to investigate the dynamic behaviour of the model for different steplengths and diffusion rates. It was seen in numerical experiments that the dynamic behaviour of measles depends on the initial distributions and the diffusion rate.

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References